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On Muskat Problems

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Abstract: In this general report, we give a description of Muskat problem an put forward Muskat problem with surface tension in the physical fact. The weak formulation for Muskat problem is given out and a relationship between Muskat problem and quasilinear hyperbolic equation is established. For Muskat problem and Muskat problem with surface tension, we prove the existence of classical solution local in time by the theory of pseudo-differential operator. Moreover, we conclude that interface stability highly depends on the mobility ration $M = \frac{\mu_0}{\mu_w}$ for Muskat problem. However, Muskat problem with surface tension is well-posed without any condition imposed on the mobility ratio M, which corresponds to the physical fact.

Part I Muskat problem

1.1 Model

When two fluids in motion occupy a porous medium, we consider a simultaneous flow of two **immiscible** fluids or phases in the pore space.

In general, **an abrupt interface** between two immiscible fluids in the macroscopic sense can't exist, i.e., there is not a continuous surface completely separating the two fluids. But the displacement of immiscible fluids is almost paratically complete. So for all practical purpose a fictitious abrupt interface may be assumed to separate the two fluids and on the each side of the surface there only exists a single phase (fluid). In oil literature, the displacement is usually called **piston-like**.

It is a free boundary problem, we call **Muskat problem**, which was proposed by Muskat in 1934 [1].

From the Law of conservation of mass and Darcy's Law, the problem (for incompressible fluids) is formulated as follows:

$$
\begin{aligned}
&-\nabla\cdot\left(\frac{k}{\mu_w}\nabla p_w\right)=0 && \text{in } Q_w(\text{water region})\\
&-\nabla\cdot\left(\frac{k}{\mu_0}\nabla p_0\right)=0 && \text{in } Q_0(\text{oil region})\\
&p_w-p_0=0 && \text{on } \Gamma(\text{interface})\\
&-\frac{k}{\mu_w}\nabla p_w\cdot n=-\frac{k}{\mu_0}\nabla p_0\cdot n=\phi v_n && \text{on } \Gamma,
\end{aligned}
$$

where k is the permeability, μ_0 and μ_w are viscosities of oil and water respectively. ϕ is the porosity, n is the normal of $\Gamma_t=\Gamma\cap\{t\}$ and v_n is the velocity of advance in the normal direction of Γ_t.

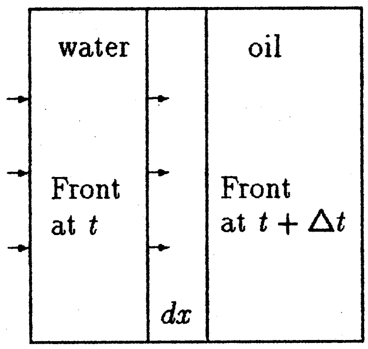
Muskat problem is a time-dependent elliptic diffraction problem with a free boundary.

Remark Muskat problem is a approximate model of an immiscible displacement. It has been assumed that an abrupt interface between two immiscible fluids is a regular surface, i.e., it has been assumed that the displacement of two fluids is **stable**. The correctness of this approximation strongly depends on the stability.

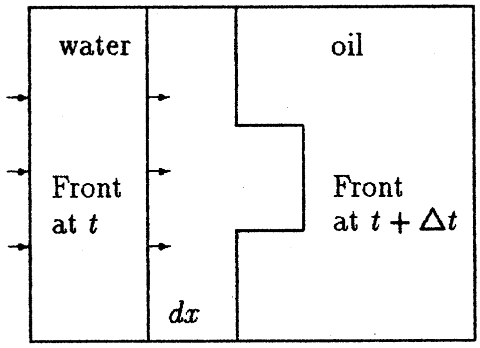
In physical fact (see [2]), interface stability depends on **the mobility ratio** $M=\frac{\mu_0}{\mu_w}$.

For a problem modelling **the extraction of oil from the ground by water**, then if $M>1$, **Instability phenomenon** (fingering phenomenon) always occurs and the displacement is instability, and if $M \leq 1$, the displacement is stable. $M \leq 1$ means $\mu_w \geq \mu_0$, i.e., for the displacement of oil by viscositied water the interface is **stable**.

In this paper, we explain that the physical fact is highly important to the well-posedness of Muskat problem.



Stable displacement



Model of finger

1.2 Weak formulation

To our knowledge, there were no essential advances yet about Muskat problem. One of the reason is that we do not know how to give its weak formulation (see [3], p.181).

Few years ago Jiang and Chen [4] found a **physical fact**: for a model which is displacement of oil by viscositied water ($M \leq 1$) and the capillary force is neglected which results in diagonal relative permeability curves, **the displacement must be piston-like**, i.e., in the above mentioned case, the Muskat problem is not only an approximate model but also an accurate model.

According to this idea, we introduce a new function s which is a saturation of water, $0 \leq s \leq 1$, and $1-s$ is a saturation of oil. Then we consider a couple system

$$(*)\quad \begin{cases} \nabla \cdot k(s)\nabla p = 0 & \text{in } Q, \\ \phi\partial_t s - \nabla \cdot k(s) f(s)\nabla p = 0 & \text{in } Q, \end{cases}$$

where $Q = Q_w \cup \Gamma \cup Q_0$, $M = \frac{\mu_0}{\mu_w}$, $k(s) = [1 + (M-1)s]$ and $f(s) = \dfrac{Ms}{1+(M-1)s}$.

The couple system $(*)$ comes from equations of motion of immiscible fluids, and the second equation in system $(*)$ is a quasilinear hyperbolic equation.

In paper [4], Jiang and Chen have proved the following

Theorem *Muskat problem and system (*) is equivalent, i.e., if* $\{p_w, p_0, \Gamma\}$ *is a solution of Muskat problem, and* Q_w, Q_0 *are regions of water and oil respectively, we suppose*

$$s(x,t) = \begin{cases} 1 & \text{in } Q_w \\ 0 & \text{in } Q_0 \end{cases} \quad \text{and} \quad p(x,t) = \begin{cases} p_w & \text{in } Q_w \\ p_0 & \text{in } Q_0. \end{cases}$$

Then $\{p, s\}$ *is a physically relevant weak solution of system* $(*)$. *Conversely, if* $\{p, s\}$ *is a weak solution of system* $(*)$, *and* $s(x,t)$ *only take two values 1 and 0,* Γ *is a discontinuous surface of* s, *which divides* Q *into two parts* Q_w *and* Q_0 *and* $s\big|_{Q_w} = 1$, $s\big|_{Q_0} = 0$, *then* $\{p\big|_{Q_w}, p\big|_{Q_0}, \Gamma\}$ *is a solution of Muskat problem.*

The relationship between Muskat problem and the theory of quasilinear hyperbolic equation is as follows:

Muskat problem	Equations of motion of immiscible fluids
Free boundary Γ	Shock wave of quasiliner hyperbolic equation
Free boundary condition $\phi v_n = -\frac{k}{\mu_w}\nabla p_w \cdot n$	Rankine-Hugoniot condition given on shock wave
$M \leq 1$	entropy condition

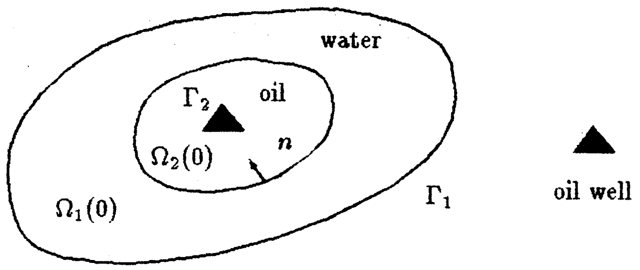
Remark System $(*)$ is a weak formulation of Muskat problem, because in $(*)$ the free boundary does not appear explicitly and they are therefore referred to as a "fixed domain" formulation.

A numerical method has been used to solve the system $(*)$. The numerical results shows that it describes clearly the process of advance of the F.B. between oil and water.

Conclusion Only $M \leq 1$, i.e., for the displacement of oil by viscositied water, we can get the weak formulation for Muskat problem.

1.3 Existence of classical solution local in time

Consider a model of extraction of oil from the ground by water as follows:



Introduce ω to be the local coordinates of points on the surface Γ_0. We also use $x = X_0(\omega)$ to denote the points on Γ_0 in $\mathbb{R}^2$. Let $n_0(\omega)$ be the unit normal to Γ_0 which is outer with respect to $\Omega_1(0)$. Let $\rho(\omega,t)$ be a function of class $C^{2,1}(\Gamma_0 \times [0,T])$ such that $\rho(\omega,0) = 0$.

Let $T > 0$ be small and let

$$\Gamma_t = \{x = X_0(\omega) + \rho(\omega,t)n_0(\omega),\ t \in [0,T]\}$$

denote the free boundary.

Straighten the free boundary (see [5]):

$$\Gamma_t \xrightarrow{\text{transformation}} \Gamma_0$$

$$\bigcup_t \Omega_i(t) \xrightarrow{\text{transformation}} Q_i = \Omega_i(0) \times [0,T] \quad \text{(i=1, 2)}$$

$$p_w(x,t),\ p_0(x,t) \xrightarrow{\text{transformation}} v_i = v_i(y,t) \qquad \text{(i=1, 2)}$$

The Muskat problem becomes

$$
(\mathrm{P}_1)\quad
\begin{cases}
L_\rho v_i(y,t) = 0 & \text{in } Q_i \ (i=1,2) \\
v_1 - v_2 = 0 & \text{on } \Gamma = \Gamma_0 \times [0,T] \\
\partial_t \rho = -k_1 S_\rho \partial_n v_1 + k_1 K_\rho \partial_w v_1 & \\
\quad = -k_2 S_\rho \partial_n v_2 + k_2 K_\rho \partial_w v_2 & \text{on } \Gamma = \Gamma_0 \times [0,T] \\
v_1 = g_1(y,t) & \text{on } \Gamma_1 \times [0,T] \\
\partial_n v_2 = g_2(y,t) & \text{on } \Gamma_2 \times [0,T]
\end{cases}
$$

where

$$
\begin{aligned}
& k_1 = \frac{k}{\phi \mu_w}, \ k_2 = \frac{k}{\mu_0} \quad \left(M = \frac{\mu_0}{\mu_w} = \frac{k_1}{k_2} \right), \\
& L_\rho = \sum_{i,j=1}^{2} a_\rho^{ij} \partial^2_{y_i y_j} + \sum_{i=1}^{2} a_\rho^i \partial_{y_i}, \\
& a_\rho^{ij} = a^{ij}(\rho, \partial_w \rho), \quad a_\rho^i = a^i(\rho, \partial_w \rho, \partial_w^2 \rho), \\
& S_\rho = a^{ij}(\rho, \partial_w \rho), \quad K_\rho = K(\rho, \partial_w \rho).
\end{aligned}
$$

Let $G \subset \mathbb{R}^n$ $(n = 1,2)$ be a bounded open domain. Define function spaces

$$
\begin{aligned}
& C_T^{k+\alpha}(\overline{G}) = C([0,T]; C^{k+\alpha}(\overline{G})), 0 < \alpha < 1, \ k = 1, 2, \cdots \\
& E^{k+\alpha, k-l+\alpha}(\overline{G}) = \left\{ v \middle| v \in C_T^{k+\alpha}(\overline{G}), \partial_t v \in C_T^{k-l+\alpha}(\overline{G}) \right\} \ (k \geq l+1, \ l \geq 1) \\
& \overset{\circ}{E}{}^{k+\alpha, k-l+\alpha}(\overline{G}) = \left\{ v \middle| v \in E^{k+\alpha, k-l+\alpha}(\overline{G}), \ v\big|_{t=0} = \partial_t v\big|_{t=0} = 0 \right\} \\
& \|v\|_{E^{k+\alpha, k-l+\alpha}} = \|v\|_{C_T^{k+\alpha}} + \|\partial_t v\|_{C_T^{k-l+\alpha}}.
\end{aligned}
$$

Suppose

(I) $\Gamma_0, \Gamma_1, \Gamma_2 \in C^{4+\alpha}$, $\ 0 < \alpha < 1$,

(II) $g_1(x,t) \in E^{4+\alpha, 3+\alpha}$, $\ g_2(x,t) \in E^{3+\alpha, 2+\alpha}$,

(III) $v_n\big|_{t=0} > 0$, i.e., the initial velocity of free boundary is positive in outer normal direction with respect to water region $\Omega_1(0)$.

(IV) $M = \frac{\mu_0}{\mu_w} = \frac{k_1}{k_2} < 1$.

Define

$$F(\rho) = \partial_t \rho + k_1 S_\rho(\omega, t)\partial_n v_1 - k_1 K_\rho(\omega, t)\partial_\omega v_1,$$

where v_1, v_2 is a solution of diffraction problem for given ρ. So the solvability of free boundary problem (P1) is equivalent to existence of a solution of the equation

$$F(\rho) = 0.$$

Theorem *Under the assumptions (I)-(IV), there exists $T_0 > 0$, such that the equation $F(\rho) = 0$ has a unique solution $\rho \in E^{2+\alpha,1+\alpha}$ $(\forall T < T_0)$.*

Outline of proof

Construct an initial approximate function $\rho_0(\omega, t)$ such that $\rho_0 \in C^{4+\alpha,2+\alpha/2}(\Gamma)$ (here $\rho_0\big|_{t=0}$, $\partial_t\rho_0\big|_{t=0}$ are given).

Define the Fréchet-derivative $F'(\rho_0)$ of nonlinear operator $F(\rho)$:

$$F'(\rho_0) : \overset{\circ}{E}{}^{2+\alpha,1+\alpha} \to \overset{\circ}{C}{}^{1+\alpha}_T \quad \text{for } \forall\delta\rho \in \overset{\circ}{E}{}^{2+\alpha,1+\alpha}$$

$$\|F(\rho_0 + \delta\rho) - F(\rho_0) - F'(\rho_0)\delta\rho\|_{C_T^{1+\alpha}} = O\big(\|\delta\rho\|^2_{E^{2+\alpha,1+\alpha}}\big).$$

It is easy to see $F'(\rho_0)\delta\rho = \partial_t(\delta\rho) + f_1(\delta\rho, \partial_\omega(\delta\rho), \delta v_1, \partial_n(\delta v_1), \partial_\omega(\delta v_1))$ and $\delta v_1, \delta v_2$ satisfy a diffraction problem as follows:

$$(\mathrm{P}_2)\quad \begin{cases} L_{\rho_0}(\delta v_i) = -(\delta L_{\rho_0})v_i & \text{in } Q_i\ (i = 1, 2) \\ \delta v_1 = \delta v_2 & \text{on } \Gamma, \\ f_1(\delta\rho, \partial_\omega(\delta\rho), \delta v_1, \partial_n(\delta v_1), \partial_\omega(\delta v_1)) & \\ \quad = f_2(\delta\rho, \partial_\omega(\delta\rho), \delta v_2, \partial_n(\delta v_1), \partial_\omega(\delta v_2)) & \text{on } \Gamma \\ \delta v_1 = 0 & \text{on } \Gamma_1 \\ \partial_n(\delta v_2) = 0 & \text{on } \Gamma_2, \end{cases}$$

where δL_{ρ_0} is the variation of the operator L_ρ at $\rho = \rho_0$.

The crucial step in our proof is to prove the invertibility of $F'(\rho_0)$. It is equivalent to prove the following:

Lemma *For any* $G \in \underset{\circ}{C}{}_T^{1+\alpha}$, the equation

$$F'(\rho_0)\delta\rho = G$$

and problem (P2) has a unique solution $(\delta\rho, \delta v_1, \delta v_2)$ such that

$$\|\delta\rho\|_{E^{2+\alpha,1+\alpha}} \leq C\|G\|_{C_T^{1+\alpha}}$$

where C depends on $\|v_i\|_{C_T^{4+\alpha}}$, $\|\partial_t v_i\|_{C_T^{2+\alpha}}$ and T.

From the lemma the existence theorem is proved by a standard way.

How to prove the lemma?

Step 1 Instead of the hyperbolic equation $F'(\rho_0)\delta\rho = G$ on a closed manifold, we consider an approximate parabolic equation

$$F'(\rho_0)\delta\rho - \varepsilon\partial_\omega^2(\delta\rho) = G$$

with periodic boundary condition $\delta\rho(\omega, t) \equiv \delta\rho(\omega + L_0, t)$ (here ω is the arc-length parameter and L_0 is the arc-length of Γ_0, $\delta\rho\big|_{t=0} = 0$).

Step 2 Using a fine "partition of unity" and freezing the coefficient at $t = 0$ and neglecting the lower order terms in each small domain.

Introducing a local coordinate we only need to consider a simplest **model problem** as follows:

$$\begin{cases}
\partial_\eta^2 w_i + \partial_\omega^2 w_i = 0 & \text{in } R_\omega^1 \times R_\eta^+ \times [0,T] \text{ and } R_\omega^1 \times R_\eta^- \times [0,T] \\
w_1 - w_2 = c_1\delta\rho \quad (c_1 > 0) & \text{on } \eta = 0 \\
k_1\partial_\eta w_1 - k_2\partial_\eta w_2 = c_2\partial_\omega(\delta\rho) & \text{on } \eta = 0 \\
\partial_t(\delta\rho) - \varepsilon\partial_\omega^2(\delta\rho) + k_1\partial_\eta w_1 + c_3\partial_\omega(\delta\rho) = G & \text{on } \eta = 0
\end{cases}$$

here transformation $\delta v_i \to w_i$ such that the equations are homogeneous.

Step 3 Taking the Fourier transformation with respect to ω and solving the resulting equations on a half line, we get

$$\widehat{\delta\rho} = \int_0^t \exp\left\{\left[-\varepsilon\xi^2 + c_1\frac{k_1k_2}{k_1+k_2}|\xi| + \left(c_3 - \frac{c_2k_1}{k_1+k_2}|\xi|\right)\xi i\right](t-\tau)\right\}\widehat{G}(\xi,\tau)d\tau$$

where $\widehat{\delta\rho}$ and $\widehat{G}$ are, respectively, the Fourier transform $\delta\rho$ and G.

Step 4 By the theory of pseudo-differential operator (see [6]) we get the estimate

$$\|\delta\rho\|_{E^{2+\alpha,1+\alpha}} \leq C\|G\|_{C_T^{1+\alpha}(\Gamma)},$$

where C does not depend on ε.

Conclusion Under the assumptions (III) and (IV), we have

$$c_1 = (-\partial_n v_1 + \partial_n v_2)\big|_{t=0} = \phi\left(\frac{1}{k_1} - \frac{1}{k_2}\right)v_n > 0.$$

So we see that only for the displacement of oil by viscositied water we may get the solvability of Muskat problem.

Part II Muskat problem with surface tension

2.1 Model

When two immiscible fluids are in contact in the interstices of a porous medium, a discontinuity in pressure exists across the interface separating them. Its magnitude depends on the interface mean curvature at the point. The difference in pressure is called **Capillary pressure** p_c (see [2]):

$$p_{nw} - p_w = p_c,$$

where p_{nw}, p_w are the pressure in nonwetting and wetting phase respectively. And from Laplace equation for capillary pressure $p_c = \sigma K$, where σ is the surface tension and K is the mean curvature.

In the piston-like displacement two immiscible fluids are in contact only at the interface which separate two fluids into two parts.

Thus Muskat problem with surface tension is formulated as follows

$$
\begin{aligned}
&-\nabla\cdot\left(\frac{k}{\mu_w}\nabla p_w\right)=0 && \text{in } Q_w\\
&-\nabla\cdot\left(\frac{k}{\mu_0}\nabla p_0\right)=0 && \text{in } Q_0\\
&p_w-p_0=\sigma k=\sigma\nabla\cdot\vec{n} && \text{on } \Gamma\\
&-\frac{k}{\mu_w}\nabla p_w\cdot\vec{n}=-\frac{k}{\mu_0}\nabla p_0\cdot\vec{n}=\phi v_n && \text{on } \Gamma.
\end{aligned}
$$

In the physical fact the capillary force may affect the stability of the front (see [2]). For the stability of the interface it is a good term, it always tend to stabilize the displacement front.

So we expect that the above Muskat problem with surface tension is well-posed without any condition imposed on the mobility ratio M.

2.2 Existence of classical solution local in time

As done in Part I, the problem can be reformulated as follows

$$
(\mathrm{P}_3)\quad
\begin{cases}
L_\rho v_i(y,t)=0 & \text{in } Q_i\ (i=1,2)\\
v_1-v_2=\sigma K(\rho,\partial_\omega\rho,\partial_\omega^2\rho) & \text{on } \Gamma,\\
\partial_t\rho=-k_1S_\rho(\rho,\partial_\omega\rho)\partial_n v_1+k_2K_\rho(\rho,\partial_\omega\rho)\partial_\omega v_1 & \\
\quad=-k_1S_\rho(\rho,\partial_\omega\rho)\partial_n v_2+k_2K_\rho(\rho,\partial_\omega\rho)\partial_\omega v_2 & \text{on } \Gamma\\
v_1=g_1(y,t) & \text{on } \Gamma_1\times[0,T]\\
\partial_n v_2=g_2(y,t) & \text{on } \Gamma_2\times[0,T]
\end{cases}
$$

Theorem *Assume $\Gamma_0\in C^{8+\alpha}$, $\Gamma_j\in C^{7+\alpha}$ $(j=1,2)$, $g_i\in E^{7+\alpha,4+\alpha}(\Gamma_j)$ $(j=1,2)$, then there exists $T_0>0$, such that the problem (P3) has a solution $\rho,v_1,v_2\in E^{4+\alpha,1+\alpha}(\Gamma_0)\times C_T^{2+\alpha}(\Omega_1(0))\times C_T^{2+\alpha}(\Omega_2(0))$ (for $T<T_0$).*

The main idea of the proof is to claim that the Fréchet derivative of the nonlinear operator

$$F(\rho) = \partial_t\rho + k_1 S_\rho(\rho, \partial_\omega\rho)\partial_n v_1 - k_1 K_\rho(\rho, \partial_\omega\rho)\partial_\omega v_1$$

is invertibility.

For this problem

$$F'(\rho_0) : \underset{0}{E}^{4+\alpha,1+\alpha}(\Gamma_0) \to C_T^{1+\alpha}(\Gamma_0).$$

Thus we have to solve the equation

$$F'(\rho_0)\delta\rho = G,$$

and get an estimate

$$\|\delta\rho\|_{E^{4+\alpha,1+\alpha}} \leq C\|G\|_{C_T^{1+\alpha}}.$$

To do this we consider a simplest **model problem** as follows:

$$\begin{cases} \partial_\eta^2 w_i + \partial_\omega^2 w_i = 0 & \text{in } R_\omega^1 \times R_\eta^+ \times [0,T] \text{ and } R_\omega^1 \times R_\eta^- \times [0,T] \\ w_1 - w_2 = -c_0\partial_\omega^2\delta\rho & \text{on } \eta = 0 \quad (c_0 > 0) \\ -k_1\partial_\eta w_1 = -k_2\partial_\eta w_2 & \text{on } \eta = 0 \end{cases}$$

and

$$\partial_t(\delta\rho) + k_1\partial_\eta w_1 + \varepsilon\partial_\omega^4(\delta\rho) = G \qquad \text{on } \eta = 0$$

Taking Fourier transformation with respect to ω, we get

$$\widehat{\delta\rho} = \int_0^t e^{(\varepsilon|\xi|^4 + b|\xi|^3)(t-\tau)}\widehat{G}(\xi,\tau)d\tau$$

where $b = \dfrac{k_2 c_0}{k_1 + k_2} > 0$, $c_0 \sim \sigma$.

By the theory of pseudo-differential operator, we get the required estimate.

Conclusion In the above model problem $c_0\partial_\omega^2\delta\rho$ $(c_0 > 0)$, which corresponds to capillary pressure $p_c = \sigma K$ $(\sigma > 0)$, is a good term. For the Muskat problem with surface tension we don't need to impose any additional conditions on mobility ratio $\frac{k_1}{k_2}$, the problem always has a solution, i.e., Muskat problem with surface tension is well-posed.

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